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Fluctuations and ergodicity of the form factor of quantum propagators and random unitary matrices

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Abstract. We consider the spectral form factor of random unitary matrices as well as of Floquet matrices of kicked tops, as given by the (squared moduli of) the traces $t_n = \text{tr } F^n$ with the integer 'time' $n = 0, \pm 1, \pm 2, \ldots$. For a typical matrix F the time dependence of the form factor $|t_n|^2$ looks erratic; only after a local time average over a suitably large time window Δn does a systematic time dependence become manifest. For matrices drawn from the circular unitary ensemble we prove ergodicity: in the limits of large matrix dimension and time window Δn the local time average has vanishingly small ensemble fluctuations and may be identified with the ensemble average. By numerically diagonalizing Floquet matrices of kicked tops with a globally chaotic classical limit we find the same ergodicity. As a byproduct we find that the traces t_n of random matrices from the circular ensembles behave very much like independent Gaussian random numbers. Again, Floquet matrices of chaotic tops share that universal behaviour. It becomes clear that the form factor of chaotic dynamical systems can be fully faithful to random-matrix theory, not only in its locally time-averaged systematic time dependence but also in its fluctuations.

1. Introduction

In the present paper we propose to discuss the time dependence of the form factor of individual dynamical systems and of individual random matrices and compare this with the ensemble-based predictions of random-matrix theory (RMT). Our investigation was motivated by recent efforts to extract universality from semiclassical arguments [2–6]. Among these, Smilansky and co-workers [5, 6] have argued that random-matrix behaviour should prevail in the autocorrelation function of the spectral determinant of dynamical systems, at least after a suitable average; the conclusions were based on assuming Gaussian statistics for the form factor fluctuations. Similarly motivated work by Prange [7] and Kunz [8] has partial overlap with ours.

In order to look for universal spectral fluctuations from a given quasi-energy spectrum one must first 'unfold' the spectrum to a constant mean level spacing, or else systematic system specific variations of the density of levels across the spectrum would be mixed up with potentially universal fluctuations. Moreover, smoothing over certain energy intervals, denoted by the overbar in $\overline{\rho(E+\Delta E)\,\rho(E)}$, for the correlation functions of the density of levels or over suitable time intervals for its Fourier transform, the form factor, is necessary. Without such averaging the form factor would display erratic variations not at all resembling the smooth behaviour predicted by RMT for averages of matrix ensembles. The level density correlator would even consist of delta peaks.

If a conservative quantum system with global chaos in its classical limit displays universal spectral fluctuations à la RMT, such universality may reveal itself in the distribution $P(s)$

of nearest-neighbour spacings, for which RMT predicts the power law $P(s) \sim s^\beta$ as $s \to 0$, and the degree β of level repulsion is determined by symmetry, most importantly the absence or presence of time-reversal invariance. Equally popular and even more easily amenable to theoretical reasoning is the two-point function of the level density $C(\Delta E) = \overline{\rho(E+\Delta E)\rho(E)} - \overline{\rho(E)}^2$, for which universal behaviour implies the same power law for small energy difference ΔE. The Fourier transform of that correlation function, the so-called form factor $K(\tau)$, may of course also be used as an indicator of universality (or of lack thereof). Ergodicity of the underlying classical dynamics [9] imparts a linear dependence on the time τ, $K(\tau) \sim |\tau|$, for $|\tau|$ larger than the shortest periods of classical periodic orbits, but smaller than the Heisenberg time τ_H (the timescale corresponding to the mean level spacing as a unit of energy). For larger times, $|\tau| \gg \tau_H$, the form factor tends to a constant value, and the character of the approach is related to the degree of level repulsion.

For the sake of concreteness we base our work on unitary $N \times N$ matrices, such as arise as Floquet matrices F for periodically driven systems with a compact phase space and thus finite-dimensional Hilbert space. As examples of such dynamics we shall take kicked tops, ones with the spherical phase space dominated by chaos and, on the other hand, symmetries chosen so as to fit any of the three principal universality classes (orthogonal, unitary and symplectic). The corresponding random-matrix ensembles are, of course, the so-called circular ones of Wigner and Dyson [13] (circular orthogonal ensemble (COE), circular unitary ensemble (CUE), circular symplectic ensemble (CSE)).

Our findings within RMT extend the ones previously presented in [11] and may be summarized as follows. They all concern the 'traces' $t_n = \mathrm{tr}\, F^n$ with $n = 1, 2, \ldots$, which give the form factor as $K(n) = |t_n|^2$; the integer exponent n serves as a discrete time, counting the number of periods of the external driving. (In the large-N limit we shall eventually introduce $\tau = n/N$ as a quasi-continuous time.) The circular ensembles yield marginal probability densities $\langle\delta(t - t_n)\rangle$ which, in the limit of large matrix dimensions, $N \to \infty$, assign Gaussian statistics to finite-order moments. In particular, the first four moments bear relations of Gaussian character, as if coming from $P(t) = (\pi\overline{|t_n|^2})^{-1}\exp(-|t|^2/\overline{|t_n|^2})$. For different traces t_n and t_m we show, again for the large-N limit, that the CUE does not give cross-correlations, in the sense $\overline{|t_n|^2|t_m|^2}/(\overline{|t_n|^2}\;\overline{|t_m|^2}) - 1 = \mathcal{O}(1/N)$ for $m \neq n$. Using these results for ensemble averages we show for the CUE that the form factor is ergodic in the large-N limit: the time average $\langle|t_n|^2/\overline{|t_n|^2}\rangle = (\Delta n)^{-1}\sum_{n'=n}^{n+\Delta n}|t_{n'}|^2/\overline{|t_n'|^2}$ has an ensemble variance vanishing in the limit of a large temporal window Δn as $1/\Delta n$. This means that with overwhelming probability every random unitary matrix drawn from the appropriate circular ensemble has the same time-averaged form factor and that the latter equals the ensemble-averaged form factor. Inasmuch as a dynamical system has a Floquet matrix typical for the appropriate ensemble one can expect universality for its time-averaged form factor as well.

By numerically diagonalizing Floquet matrices of kicked tops from the various universality classes we have calculated, for each of these, the form factor $K(n) = |t_n|^2$ and its time average over a finite window Δn. Normalizing to the ensemble-averaged form factor $\overline{|t_n|^2}$ à la RMT we throw all $|t_n|^2/\overline{|t_n|^2}$ for a given top into a histogram and find this to reproduce the Gaussian behaviour predicted by RMT. The ergodic character of random matrices is also respected in full by the Floquet matrices of chaotic tops: the time averages $\langle|t_n|^2/\overline{|t_n|^2}\rangle$ come out to have variances within the respective data sets $\{t_n\}$ varying with the time window Δn as $1/\Delta n$.

2. The form factor and its fluctuations in ensembles of random matrices

The density of eigenvalues $\mathrm{e}^{-\mathrm{i}\varphi_i}$ of a unitary matrix F can be written as

$$\rho(\varphi) = \frac{1}{2\pi N} \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\varphi-\varphi_i)} = \frac{1}{2\pi N} \sum_{n=-\infty}^{\infty} t_n \mathrm{e}^{\mathrm{i}n\varphi}. \tag{1}$$

We herein meet the traces

$$t_n = \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}n\varphi_i} = \mathrm{tr}\, F^n \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{2}$$

If F is drawn from a homogeneous ensemble of random matrices like any of the familiar circular ones of Wigner and Dyson, one has the ensemble average $\overline{t_n t_{n'}} = \delta_{n,-n'}\overline{|t_n|^2}$ and thus the two-point correlation function

$$\begin{aligned} C(e) &= \overline{\rho(\varphi + e2\pi/N)\rho(\varphi)} - \overline{\rho}^2 \\ &= \sum_{n\neq 0} \overline{|t_n|^2} \mathrm{e}^{\mathrm{i}2\pi ne/N}. \end{aligned} \tag{3}$$

The mean form factor $\overline{|t_n|^2}$ is well known for the circular ensembles [11, 14] and will be given further below. Since we want to determine higher-order moments like $\overline{|t_n|^4}$, it is best to consider the characteristic function

$$\begin{aligned} \tilde{P}^{\beta}_{nN}(k) &= \overline{\exp\left(-\frac{\mathrm{i}}{2} \sum_i (k\mathrm{e}^{-\mathrm{i}n\varphi_i} + k^*\mathrm{e}^{\mathrm{i}n\varphi_i}) \right)} \\ &= \overline{\prod_{i=1}^{N} \exp(-\mathrm{i}|k| \cos(n\varphi_i))} \end{aligned} \tag{4}$$

where the joint densities of eigenphases of the circular ensembles [11, 14] may be used to calculate the averages for $\beta = 1$ (COE), $\beta = 2$ (CUE) and $\beta = 4$ (CSE). The homogeneity of these ensembles entails the characteristic function to depend on k and k^* only through the modulus $|k|$ which we shall simply denote by k henceforth. It follows that all odd-order moments vanish, as well as those even-order ones where a trace t_n is not accompanied by its complex conjugate t_n^* to form powers of $|t_n|^2$. The non-vanishing moments are

$$\overline{|t_n|^{2m}} = (-4)^m \binom{2m}{m}^{-1} \frac{\mathrm{d}^{2m}}{\mathrm{d}k^{2m}} \tilde{P}^{\beta}_{nN}(k)|_{k=0}. \tag{5}$$

3. CUE

The characteristic function is easily evaluated using any of the standard methods of RMT and takes the form of a Toeplitz determinant,

$$\tilde{P}^{2}_{nN}(k) = \det M \qquad M_{\mu\nu} = \sum_{s=-\infty}^{\infty} (-\mathrm{i})^{|s|} J_{|s|}(k) \delta_{\mu,\nu-ns} \tag{6}$$

with J_s the Bessel function of integer order. The desired Taylor coefficients of $\det M(k)$ are most conveniently evaluated through the expansions

$$\begin{aligned} M(k) &= 1 + kM^{(1)} + \frac{k^2}{2} M^{(2)} + \frac{k^3}{3!} M^{(3)} + \cdots \\ \det M(k) &= \mathrm{e}^{\mathrm{tr}\ln M(k)} = \exp\left(\frac{k^2}{2}\tau_2 + \frac{k^4}{4!}\tau_4 + \cdots \right). \end{aligned} \tag{7}$$

The derivatives $M^{(i)} = \mathrm{d}^i M(k)/\mathrm{d}k^i|_{k=0}$ can be calculated with the help of $J_s(0) = \delta_{s,0}$ and $\mathrm{d}J_s(k)/\mathrm{d}k = (J_{s-1}(k) - J_{s+1}(k))/2$ as

$$\begin{aligned} M^{(1)}_{\mu\nu} &= -\frac{\mathrm{i}}{2}(\delta_\mu^{\nu-n} + \delta_\mu^{\nu+n}) \\ M^{(2)}_{\mu\nu} &= -\tfrac{1}{4}(\delta_\mu^{\nu-2n} + \delta_\mu^{\nu+2n}) - \tfrac{1}{2}\delta_\mu^\nu \\ M^{(3)}_{\mu\nu} &= \frac{\mathrm{i}}{8}(\delta_\mu^{\nu-3n} + \delta_\mu^{\nu+3n}) + \frac{3\mathrm{i}}{8}(\delta_\mu^{\nu-n} + \delta_\mu^{\nu+n}) \\ M^{(4)}_{\mu\nu} &= \tfrac{1}{16}(\delta_\mu^{\nu-4n} + \delta_\mu^{\nu+4n}) + \tfrac{1}{4}(\delta_\mu^{\nu-2n} + \delta_\mu^{\nu+2n}) + \tfrac{3}{8}\delta_\mu^\nu. \end{aligned} \tag{8}$$

Inasmuch as the Taylor coefficients of $\det M(k)$ are related to the moments $\overline{|t_n|^{2m}}$ and the Taylor coefficients of its logarithm $\ln\det M = \mathrm{tr}\ln M$ to the corresponding cumulants, their interrelations

$$\begin{aligned} \tau_1 &= \mathrm{tr}\, M^{(1)} \\ \tau_2 &= \mathrm{tr}(M^{(2)} - M^{(1)2}) \\ \tau_3 &= \mathrm{tr}(M^{(3)} + 2M^{(1)3} - 3M^{(1)}M^{(2)}) \\ \tau_4 &= \mathrm{tr}(M^{(4)} + 12M^{(1)2}M^{(2)} - 6M^{(1)4} - 3M^{(2)2} - 4M^{(1)}M^{(3)}) \end{aligned} \tag{9}$$

resemble the familiar ones between moments and cumulants of random variables. One arrives at $\tau_1 = \tau_3 = 0$ and

$$\tau_2 = \begin{cases} -n/2 & 0 < n \leqslant N \\ -N/2 & N \leqslant n \end{cases} \tag{10}$$

$$\tau_4 = \begin{cases} 0 & 0 < n \leqslant N/2 \\ \tfrac{3}{8}N - \tfrac{3}{4}n & N/2 \leqslant n \leqslant N \\ -\tfrac{3}{8}N & N \leqslant n \end{cases} \tag{11}$$

and these give the well known CUE form factor

$$\overline{|t_n|^2} = \begin{cases} n & 0 < n \leqslant N \\ N & N \leqslant n \end{cases} \tag{12}$$

as well as its variance

$$\mathrm{Var}(|t_n|^2) = \overline{|t_n|^4} - \overline{|t_n|^2}^2 = 4\tau_2^2 + \tfrac{8}{3}\tau_4 \tag{13}$$

$$= \begin{cases} n^2 & 0 < n \leqslant N/2 \\ n^2 - 2n + N & N/2 \leqslant n \leqslant N \\ N^2 - N & N \leqslant n \end{cases} \tag{14}$$

$$= \overline{|t_n|^2}^2 + \overline{|t_{2n}|^2} - 2\overline{|t_n|^2}. \tag{15}$$

Within the interval $0 < n \leqslant N/2$ the mean $\overline{|t_n|^2}$ and the variance $\mathrm{Var}(|t_n|^2)$ are related as if t_n had a Gaussian distribution

$$P(t_n) = \frac{1}{\pi\overline{|t_n|^2}}\mathrm{e}^{-|t_n|^2/\overline{|t_n|^2}} \tag{16}$$

and that distribution also correctly reflects the vanishing of all odd-order moments and cumulants. In [11] we had established the more general result that (16) even reproduces all moments $\overline{|t_n|^{2m}}$ with $n \leqslant N/m$. Below we shall show that the Gaussian relation between $\overline{|t_n|^2}$ and $\mathrm{Var}(|t_n|^2)$ prevails even for $n > N/2$, up to asymptotically negligible corrections of order $\ln(N)/N$.

4. COE and CSE

For the COE and CSE the evaluation of $P^{\beta}_{nN}(k)$ in (4) proceeds analogously [1] and yields the Pfaffians $P^{\beta}_{nN}(k) = \sqrt{\det A^{\beta}(k)}$,

$$\begin{aligned} A^1_{\underline{m}\underline{m}'}(k) &= \sum_{s,s'} J_{|s|}(k)\, J_{|s'|}(k) \frac{(-\mathrm{i})^{|s|+|s'|}}{\underline{m}+ns} \delta^{-\underline{m}-n(s+s')}_{\underline{m}} \\ A^4_{\underline{m}\underline{m}'}(k) &= (\underline{m}-\underline{m}') \sum_s J_{|s|}(2k)(-\mathrm{i})^{|s|} \delta^{-\underline{m}-ns}_{\underline{m}} \end{aligned} \tag{17}$$

wherein the underlined indices $\underline{m}, \underline{m}'$ are semi-integer and in the ranges $|\underline{m}|, |\underline{m}'| \leqslant (N-1)/2$ for $\beta = 1$ and $|\underline{m}|, |\underline{m}'| \leqslant (2N-1)/2$ for $\beta = 4$. One encounters more complicated expressions for the expansion of $\ln A^{\beta}(k)$ since $A^{\beta}(0) \neq 1$ and an additional factor $\frac{1}{2}$ in the τ_i due to the square root in the Pfaffian. To present our results in a concise form we introduce the shorthands

$$f^b_a = \sum_{m=a+1}^{b} \frac{1}{m+n-(N+1)/2} \qquad g^b_a = \sum_{m=a+1}^{b} \frac{1}{(m+n-(N+1)/2)^2}. \tag{18}$$

For the COE we find the form factor

$$\overline{|t^{\mathrm{COE}}_n|^2} = \begin{cases} 2n - nf^N_{N-n} & 0 < n \leqslant N \\ 2N - nf^N_0 & N \leqslant n \end{cases} \tag{19}$$

and the variance

$$\mathrm{Var}(|t^{\mathrm{COE}}_n|^2) - \overline{|t^{\mathrm{COE}}_n|^2}^2 = \begin{cases} 8n(f^N_{-n} - \frac{1}{2}f^N_{N-n} - \frac{1}{4}f^{N+n}_{N-n}) - 2n^2 g^N_{N-n} \\ \qquad 0 < n \leqslant N/2 \\ 8n(f^N_{-n} + f^n_{-n} - \frac{1}{2}f^N_{N-n} - \frac{1}{4}f^{N+n}_{N-n}) - 2n^2 g^N_{N-n} + 8N - 16n \\ \qquad N/2 \leqslant n \leqslant N \\ 8n(f^N_0 + \frac{1}{4}f^n_0 - \frac{1}{4}f^{N+n}_N) - 2n^2 g^N_0 - 8N \\ \qquad N \leqslant n. \end{cases} \tag{20}$$

For the CSE, i.e. the case $\beta = 4$, Kramers' degeneracy is present. It suffices to take every eigenvalue into account once: assuming the matrix to be of size $2N$ we work with the 'trace' $t^{\mathrm{CSE}}_n = \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\varphi_i} = \frac{1}{2} \operatorname{tr} F^n$ and thus account for every Kramers' doublet once.

The form factor and its variance are obtained as

$$\overline{|t^{\mathrm{CSE}}_n|^2} = \begin{cases} n/2 - \dfrac{n}{4} f^{-N/2}_{-n-N/2} & 0 < n \leqslant 2N \\ N & 2N \leqslant n \end{cases} \tag{21}$$

$$\mathrm{Var}(|t^{\mathrm{CSE}}_n|^2) - \overline{|t^{\mathrm{CSE}}_n|^2}^2 = \begin{cases} -\dfrac{n^2}{8} g^{-N/2}_{-n-N/2} + \dfrac{n}{8}(f^{-N/2}_{-n-N/2} - f^{n-N/2}_{-N/2}) & 0 < n \leqslant N \\ -\dfrac{n^2}{8} g^{-n+3/2N}_{-N/2} + \dfrac{n}{4} f^{-N/2}_{-n-N/2} + N - n & N \leqslant n \leqslant 2N \\ -N & 2N \leqslant n. \end{cases} \tag{22}$$

As the important result we find that fluctuations are of the same order as the form factor itself. We therefore cannot expect an individual Floquet or random matrix to yield a sequence $|t_n|^2$ in accordance with the ensemble mean without any averaging. We shall presently show that the form factor is ergodic, i.e. the ensemble mean is approached by taking a local time average in Pandey's sense [10]. Before proceeding to that endeavour we consider the fluctuations in the limit $N \to \infty$, where they simplify significantly.

5. Asymptotic fluctuations

Prior to taking fluctuations to the limit $N \to \infty$ a normalization is in order. It seems quite natural to refer the discrete time n to N, which is the Heisenberg time for $\beta = 1, 2$ and half of the Heisenberg time for $\beta = 4$. We write $\tau = n/N$, imagine τ kept fixed as $N \to \infty$, and eventually allow τ to range among the real numbers. The asymptotic form factors arising in that limit are of course well known.

No simplification arises in the unitary case,

$$\frac{\overline{|t_n|^2}}{N} \to K_{\mathrm{CUE}}(\tau) = \begin{cases} |\tau| & 0 < |\tau| \leqslant 1 \\ 1 & 1 \leqslant |\tau|. \end{cases} \tag{23}$$

The CUE form factor is everywhere continuous but its first derivative jumps at $\tau = 0$ and at the Heisenberg time, $\tau = \pm 1$.

In the orthogonal and symplectic cases the sums f_a^b and g_a^b must be turned into integrals and obviously yield logarithms. The COE form factor then becomes

$$\frac{\overline{|t_n|^2}}{N} \to K_{\mathrm{COE}}(\tau) = \begin{cases} 2|\tau| - |\tau| \ln(2|\tau| + 1) & 0 < |\tau| \leqslant 1 \\ 2 - |\tau| \ln\left(\dfrac{2|\tau| + 1}{2|\tau| - 1}\right) & 1 \leqslant |\tau|. \end{cases} \tag{24}$$

We still encounter a jump of $K'_{\mathrm{COE}}(\tau)$ at $\tau = 0$ but at the Heisenberg time $\tau = \pm 1$ a jump arises only for the third derivative.

The asymptotic CSE form factor,

$$\frac{\overline{|t_n|^2}}{N} \to K_{\mathrm{CSE}}(\tau) = \begin{cases} \dfrac{|\tau|}{2} - \dfrac{|\tau|}{4} \ln(|1 - |\tau||) & 0 < |\tau| \leqslant 2 \\ 1 & 2 \leqslant |\tau| \end{cases} \tag{25}$$

displays a logarithmic singularity at $\tau = \pm 1$, i.e. at half the Heisenberg time. Jumps arise in $K'_{\mathrm{CSE}}(\tau)$ at $\tau = 0$ and in $K'''_{\mathrm{CSE}}(\tau)$ at the Heisenberg time $\tau = \pm 2$.

As for the variances $\mathrm{Var}(|t_n|^2/N)$ we find strictly Gaussian behaviour in the limit $N \to \infty$,

$$\lim_{N\to\infty} \mathrm{Var}\left(\frac{|t_n|^2}{N}\right) = \lim_{N\to\infty} \left(\frac{\overline{|t_n|^2}}{N}\right)^2 = K(\tau)^2 \tag{26}$$

for all three circular ensembles. A little side remark is indicated, however, for the case of the CSE. We here find a funny correction to the Gaussian behaviour which according to taste one might keep or drop,

$$\lim_{N\to\infty} \mathrm{Var}\left(\frac{|t_n|^2}{N}\right) - K_{\mathrm{CSE}}(\tau)^2 = -\delta_{|\tau|,1} \frac{\pi^2}{16} \tag{27}$$

where $\delta_{|\tau|,1}$ denotes the Kronecker delta. In as much as the rhs for $|\tau| = 1$ (at half the Heisenberg time) is of order unity one would want to keep the correction and even feast on its non-Gaussian character; on the other hand, if one wants to look at τ as a continuous variable the exceptional points $\tau = \pm 1$ accomodate removable singularities without weight for integrals. Furthermore, the rhs of (27) is relatively small since $K_{\mathrm{CSE}}(\tau)$ diverges logarithmically at $\tau = 1$. Figure 1 gives an impression for the approach of $\mathrm{Var}(|t_n|^2/N) - (\overline{|t_n|^2}/N)^2$ to the asymptotic $-\delta_{|\tau|,1}\pi^2/16$ as N grows. If one wants to study that approach analytically, one must isolate in $\mathrm{Var}(|t_n|^2/N) - (\overline{|t_n|^2}/N)^2$ the term which does not manifestly vanish for $N \to \infty$ at least as $\ln(N)/N$. With the help of (21) and (22) one finds that term as

$$h(n, N) = \begin{cases} -\dfrac{n^2}{8N^2} g_0^N & 0 < \tau \leqslant 1 \\ -\dfrac{n^2}{8N^2} g_n^{2N} & 1 \leqslant \tau \leqslant 2 \end{cases} \tag{28}$$

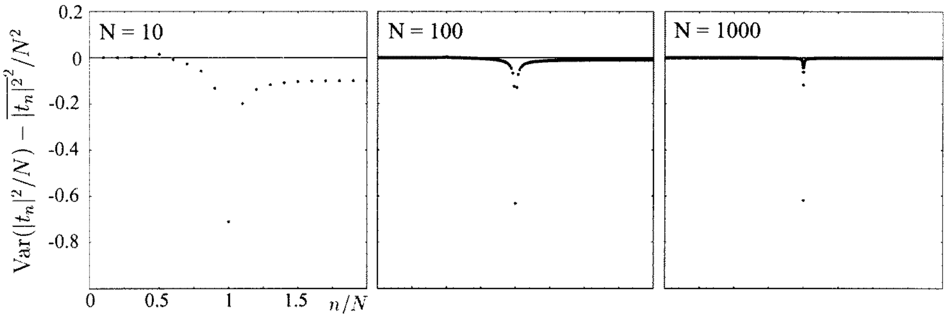

Figure 1. The approach of the symplectic ensemble's $\mathrm{Var}(|t_n|^2/N) - (\overline{|t_n|^2/N})^2$ to the asymptotic $-\delta_{|\nu|,1}\pi^2/16$ with growing N at half of the Heisenberg time.

which can be expressed in terms of the polygamma function $\psi^{(1)}$, the first derivative of the digamma function, as

$$h(n, N) = \begin{cases} -\dfrac{n^2}{8N^2}(\psi^{(1)}(N - n + \frac{1}{2}) - \psi^{(1)}(N + \frac{1}{2})) & 0 < n \leqslant N \\ -\dfrac{n^2}{8N^2}(\psi^{(1)}(n - N + \frac{1}{2}) - \psi^{(1)}(N + \frac{1}{2})) & N \leqslant n \leqslant 2N. \end{cases} \tag{29}$$

The polygamma function $\psi^{(1)}$ has the familiar integral representation

$$\psi^{(1)}(z) = \int_0^\infty \mathrm{d}t \, \frac{t}{1 - \mathrm{e}^{-t}} \mathrm{e}^{-zt}. \tag{30}$$

We need the foregoing representation for $z = N(1 + 1/(2N))$, $z = N(1 - \tau + 1/(2N))$ and $z = N(\tau - 1 + 1/(2N))$. Obviously now, for $\tau \neq 1$ fixed and $N \to \infty$ we have $\psi^{(1)} \to 0$ in all terms involved; at $\tau = 1$, however, the limit $N \to \infty$ yields $\psi^{(1)}(\frac{1}{2}) = \pi^2/2$ and thus $h(N, N) = -\pi^2/16$. The decay of $h(\tau N, N)$ to zero for $N \to \infty$ at fixed $\tau \neq 1$ is found from the asymptotic large-z behaviour of the polygamma function

$$\psi^{(1)}(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{31}$$

as

$$h(\tau N, N) = \begin{cases} -\dfrac{1}{8N}\dfrac{\tau^3}{1 - \tau} & 0 < \tau < 1 \\ -\dfrac{1}{8N}\dfrac{\tau^3}{\tau - 1} & 1 < \tau \leqslant 2. \end{cases} \tag{32}$$

6. Ergodicity of the form factor

Fluctuations of the form factor tend to be suppressed by a local time average,

$$\langle |t_n|^2 \rangle = \frac{1}{\Delta n} \sum_{n'=n-\Delta n/2}^{n+\Delta n/2} |t_{n'}|^2. \tag{33}$$

As one increases the time window Δn one expects the time average to become equivalent to the ensemble average. Inasmuch as the systematic time dependence should not be washed out by the time average, one must first let $N \to \infty$ and subsequently $\Delta n \to \infty$.

To uncover the expected ergodicity we consider the form factor as normalized to its ensemble mean, $|t_n|^2/\overline{|t_n|^2}$. We realize that $\langle|t_n|^2/\overline{|t_n|^2}\rangle = 1$ holds trivially and propose to show that the ensemble variance of the temporal mean vanishes, $\mathrm{Var}(\langle|t_n|^2/\overline{|t_n|^2}\rangle) \to 0$, as $\Delta n \to \infty$. The variance in question is defined as

$$\mathrm{Var}(\langle|t_n|^2/\overline{|t_n|^2}\rangle) = \left(\frac{1}{\Delta n}\right)^2 \sum_{n',n''=n-\Delta n/2}^{n+\Delta n/2} \left(\frac{\overline{|t_{n'}|^2|t_{n''}|^2}}{\overline{|t_{n'}|^2}\;\overline{|t_{n''}|^2}} - 1\right). \tag{34}$$

Its evaluation requires knowledge of the cross-correlator $\overline{|t_{n'}|^2|t_{n''}|^2}$ for $n' \neq n''$. We may determine the cross-correlator from the characteristic function

$$\tilde{P}(k_{n'}, k_{n''}) = \overline{\exp\left(-\frac{\mathrm{i}}{2}\sum_i((k_{n'}\mathrm{e}^{-\mathrm{i}n'\varphi_i} + k_{n''}\mathrm{e}^{-\mathrm{i}n''\varphi_i}) + \text{c.c.})\right)} \tag{35}$$

which generalizes the single-trace one (4) and yields

$$\overline{|t_{n'}|^2|t_{n''}|^2} = (2\mathrm{i})^4 \frac{\partial^4}{\partial k_{n'}\partial k_{n'}^*\partial k_{n''}\partial k_{n''}^*}\tilde{P}(k_{n'}, k_{n''})\bigg|_0. \tag{36}$$

Once more employing the technique used to establish the single-trace characteristic function one finds for the CUE

$$\begin{aligned}&\tilde{P}^2(k_{n'}, k_{n''}) = \det M\\ &M_{\mu\nu} = \int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\mathrm{e}^{\mathrm{i}(\mu-\nu)\phi}\exp\left(-\frac{\mathrm{i}}{2}(k_{n'}\mathrm{e}^{-\mathrm{i}n'\phi} + k_{n''}\mathrm{e}^{-\mathrm{i}n''\phi} + \text{c.c.})\right)\end{aligned} \tag{37}$$

and from here $(n' > n'')$

$$\frac{\overline{|t_{n'}|^2|t_{n''}|^2}}{\overline{|t_{n'}|^2}\;\overline{|t_{n''}|^2}} - 1 = \begin{cases}\dfrac{N-n'-n''}{n'n''} & n' < N, n'+n'' > N\\ -\dfrac{N-n'+n''}{Nn''} & N < n', n'-n'' < N\\ 0 & \text{otherwise.}\end{cases} \tag{38}$$

Upon inserting these cross-correlations into (34) the desired variance is obtained. An upper bound for this variance is found by neglecting all negative contributions to the sum, i.e. all terms with $n' \neq n''$. The estimate then reads

$$\mathrm{Var}(\langle|t_n|^2/\overline{|t_n|^2}\rangle) \leqslant \frac{1}{\Delta n} \stackrel{\Delta n\to\infty}{\longrightarrow} 0. \tag{39}$$

The asserted ergodicity of the form factor is thus proven.

7. Asymptotic independence of the traces

With the cross-correlations (38) at hand we can now show that the traces $t_{n'}, t_{n''}$ with $n' \neq n''$ are asymptotically uncorrelated in the limit $N \to \infty$. Again we assume $n' > n''$ and first consider the case $n' < N$, $n' + n'' > N$ in (38), where we can estimate

$$\left|\frac{N-n'-n''}{n'n''}\right| < \frac{n''}{n'n''} = \frac{1}{n'} \sim \frac{1}{N}. \tag{40}$$

In this situation n' cannot be small compared with N since $n' > n''$ and $n' + n'' > N$. In the case $N < n', n' - n'' < N$ the estimate reads

$$\left|\frac{N-n'+n''}{Nn''}\right| < \frac{n''}{Nn''} = \frac{1}{N}. \tag{41}$$

We see that the cross-correlations (38) vanish no more slowly than $1/N$ for $N \to \infty$.

8. The form factor of the kicked top

The Floquet operator of the kicked top with a chaotic classical limit has proven faithful to RMT predictions in many aspects [1, 12]. Among these, the level-spacing distribution and the integrated two-point correlator are noteworthy. We here propose to reveal faithfulness of the top to RMT for the form factor and its fluctuations.

The Floquet operator is built of angular momentum operators J_x, J_y, J_z and can be understood as a succession of linear rotations and torsions. For integer angular momentum

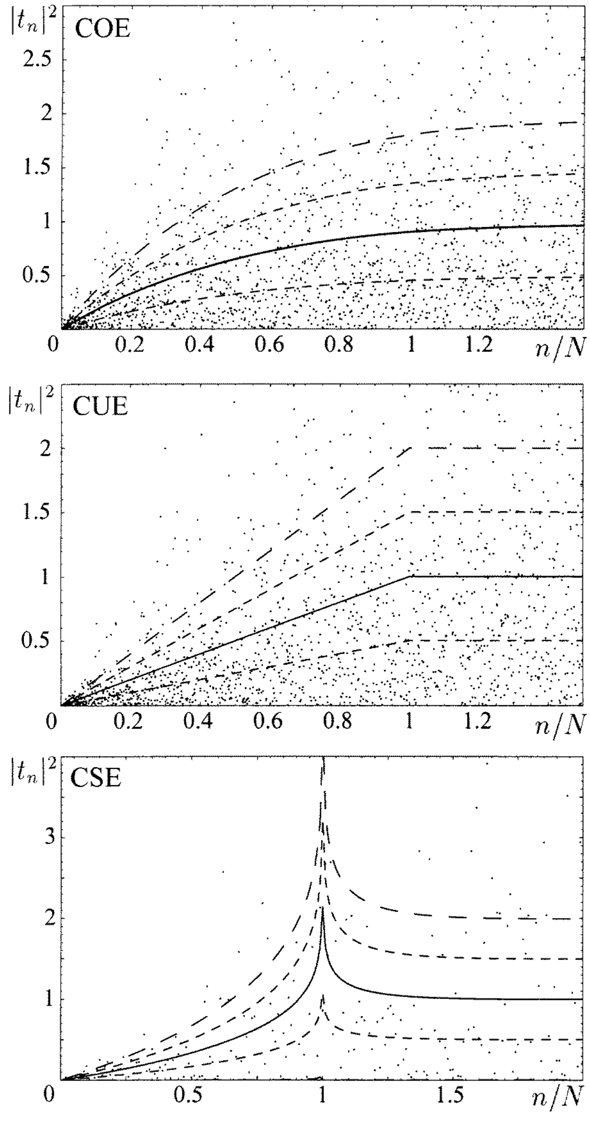

Figure 2. Form factors of the orthogonal, unitary and symplectic Floquet operators with parameters as given in the text. Also shown are the respective ensemble means and stripes of one and two linear variances width around the mean.

values j the operator to be considered is of the form

$$F = \mathrm{e}^{-\mathrm{i}\frac{\tau_z}{2j+1}J_z^2 - \mathrm{i}\beta_z J_z}\mathrm{e}^{-\mathrm{i}\beta_y J_y}\mathrm{e}^{-\mathrm{i}\frac{\tau_x}{2j+1}J_x^2 - \mathrm{i}\beta_x J_x} \tag{42}$$

and depending on the rotation angles β_i and torsion constants τ_i belongs to the orthogonal ($\tau_z = 10$, $\beta_z = 1$, $\beta_y = 1$, $\tau_x = \beta_x = 0$) or unitary ($\tau_z = 10$, $\beta_z = 1$, $\beta_y = 1$, $\tau_x = 4$, $\beta_x = 1.1$) universality class. For semi-integer j the Floquet operator

$$\begin{aligned} F &= \mathrm{e}^{-\mathrm{i}(\frac{\tau_1}{2j+1}J_z^2 + \frac{\tau_2}{2j+1}(J_x J_z + J_z J_x) + \frac{\tau_3}{2j+1}(J_x J_y + J_y J_x))}\mathrm{e}^{-\mathrm{i}\frac{\tau_4}{2j+1}J_z^2} \\ \tau_1 &= 1 \qquad \tau_2 = 4 \qquad \tau_3 = 2.1 \qquad \tau_4 = 10 \end{aligned} \tag{43}$$

lacks all geometric symmetries and thus belongs to the symplectic universality class [1]. For numerical work F is represented as a matrix of dimension $N = 2j + 1$ in the basis of eigenvectors of J_z, $J_z|jm\rangle = m|jm\rangle$, with fixed j and $-j \leqslant m \leqslant +j$.

Figure 2 shows the form factors for the three cases chosen and reveals wild fluctuations about the ensemble means which are displayed as the solid curves. The dashed curves together with the abscissa border stripes of one and two asymptotic linear variances around the means. Gaussian distributions for the t_n would predict the fractions $(1 - 1/\mathrm{e})/\sqrt{(\mathrm{e})} \approx 0.383$ and $(1 - 1/\mathrm{e}^2) \approx 0.865$ of all traces to lie within the one-variance and the two-variances stripes, respectively. On counting we find the fractions to be 0.390 and 0.875 for the orthogonal top, 0.397 and 0.870 for the unitary one, and finally 0.386 and 0.858 for the symplectic top. The Gaussian expectation is borne out well by these numbers and further substantiated by the histograms in figure 3 for the normalized moduli $\tau_n = |t_n|/\sqrt{\overline{|t_n|^2}}$. Gaussian statistics would

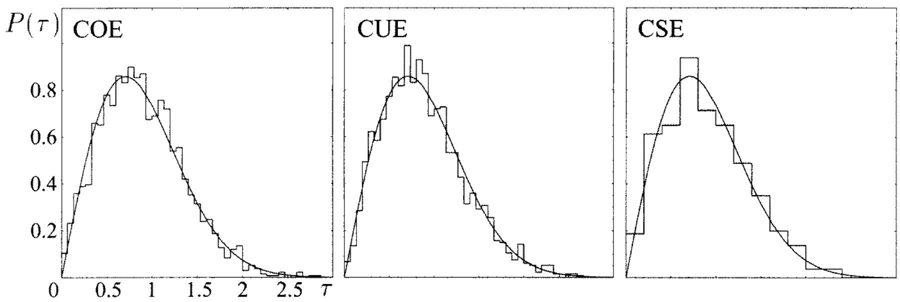

Figure 3. The distributions of the Floqet operators' normalized traces $\tau_n = |t_n|/\sqrt{\overline{|t_n|^2}}$ displayed by the histograms show good agreement with the distribution $2\tau\mathrm{e}^{-\tau^2}$ (smooth curve) expected for Gaussian statistics.

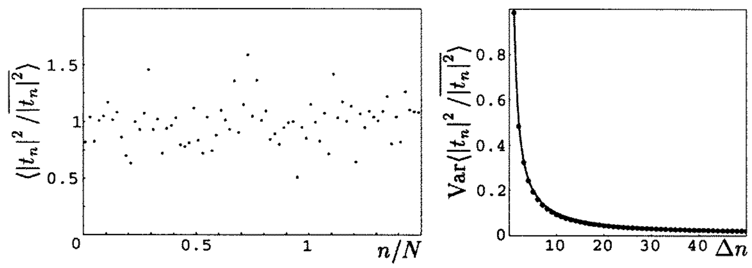

Figure 4. Left: form factor of the unitary top smoothened by a local time average over the window $\Delta n = 20$. Right: width of the fluctuating band for $|t_n|^2/\overline{|t_n|^2}$ as a function of the time window Δn (dots) together with the $1/\Delta n$ bound (curve) predicted by RMT.

yield the distribution $2\tau e^{-\tau^2}$ which is displayed as the smooth curve and well approximated by the histograms.

Finally, we illustrate ergodicity of the form factor of the top (unitary variant) in figure 4. The left part shows the form-factor fluctuations after a local time average over the window $\Delta n = 20$. Comparison with figure 2 reveals the smoothing effect of the local time average. We have also evaluated the variance of the 'band' of points $|t_n|^2/\overline{|t_n|^2}$ as a function of the time window Δn. The result is represented by the dots in the right part of figure 4, together with the $1/\Delta n$ bound predicted by RMT. Again, the faithfulness of the top to RMT is impressive.

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